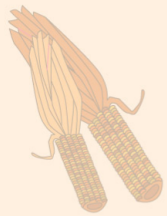
4.2 Area



Corn

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the **ith term** of the sum, and the **upper and lower bounds of summation** are n and 1.

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In Exercises 1–6, find the sum. Use the summation capabilities of a graphing utility to verify your result.

1. $\sum_{i=1}^{4}(2i + 1)$

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In Exercises 7–14, use sigma notation to write the sum.

11. $\left[\left(\frac{2}{n}\right)^3 - \frac{2}{n}\right]\left(\frac{2}{n}\right) + \cdots + \left[\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right]\left(\frac{2}{n}\right)$

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THEOREM 4.2 Summation Formulas

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

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In Exercises 15–20, use the properties of summation and Theorem 4.2 to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

20. $\sum_{i=1}^{10} i(i^2 + 1)$

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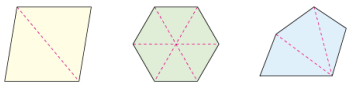
Area

In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is $A = bh$, as shown in Figure 4.5, it is actually more proper to say that this is the *definition* of the **area of a rectangle.**

From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 4.6. Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 4.7.



Parallelogram Hexagon Polygon

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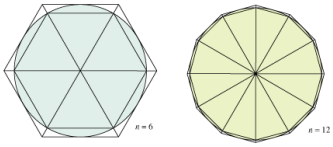

The exhaustion method for finding the area of a circular region

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The Area of a Plane Region

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Upper and Lower Sums

The procedure used in Example 3 can be generalized as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$, as shown in Figure 4.10. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.

To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, as shown in Figure 4.11. The endpoints of the intervals are as follows.

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdots < \overbrace{a + n(\Delta x)}^{x_n = b}$$

Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$f(m_i)$ = Minimum value of $f(x)$ in ith subinterval

$f(M_i)$ = Maximum value of $f(x)$ in ith subinterval

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each* i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\left(\begin{array}{c}\text{Area of inscribed}\\ \text{rectangle}\end{array}\right) = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \left(\begin{array}{c}\text{Area of circumscribed}\\ \text{rectangle}\end{array}\right)$$

The sum of the areas of the inscribed rectangles is called a **lower sum,** and the sum of the areas of the circumscribed rectangles is called an **upper sum.**

Lower sum $= s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x$ Area of inscribed rectangles

Upper sum $= S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x$ Area of circumscribed rectangles

From Figure 4.12, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

$s(n) \le$ (Area of region) $\le S(n)$

Area of inscribed rectangles is less than area of region. Area of region. Area of circumscribed rectangles is greater than area of region.

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In Exercises 27–30, use upper and lower sums to approximate the area of the region using the indicated number of subintervals (of equal width).

30. $y = \sqrt{1 - x^2}$

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In Exercises 39–44, find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

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THEOREM 4.3 Limit of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta x$$

$$= \lim_{n\to\infty} S(n)$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

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Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \qquad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$ (see Figure 4.14).

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In Exercises 47–56, use the limit process to find the area of the region between the graph of the function and the x-axis over the indicated interval. Sketch the region.

	Function	*Interval*
52.	$y = 1 - x^2$	$[-1, 1]$

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In Exercises 57–62, use the limit process to find the area of the region between the graph of the function and the y-axis over the indicated y-interval. Sketch the region.

58. $g(y) = \frac{1}{2}y, 2 \le y \le 4$

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In Exercises 63–66 use the *Midpoint Rule*

Area $\approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$

with $n = 4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the indicated interval.

Function	*Interval*
65. $f(x) = \tan x$	$\left[0, \frac{\pi}{4}\right]$

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