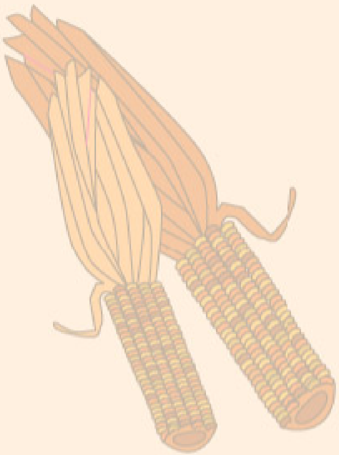
4.3 Riemann Sums and Definite Integrals



Riemann Sums

In the definition of area given in Section 4.2, the partitions have subintervals of *equal width*. This was done only for computational convenience. We begin this section with an example that shows that it is not necessary to have subintervals of equal width.

Definition of a Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\, \Delta x_i, \qquad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ.

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b - a}{n}.$$

Regular partition

Definition of a Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

It is not a coincidence that the notation for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when we discuss the Fundamental Theorem of Calculus. For now it is important to see that definite integrals and indefinite integrals are different identities. A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.

A sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this text.

THEOREM 4.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

In Exercises 9–12, express the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

12. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right) \Delta x_i \qquad [1, 3]$

THEOREM 4.5 The Definite Integral as the Area of a Region

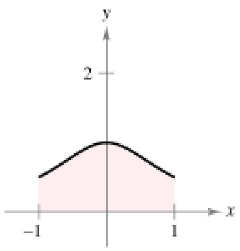
If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

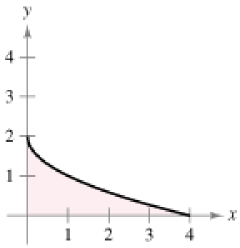
(See Figure 4.22.)

In Exercises 13–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

18. $f(x) = \dfrac{1}{x^2 + 1}$



22. $f(y) = (y - 2)^2$



In Exercises 23–32, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral ($a > 0, r > 0$).

28. $\int_0^8 (8 - x)\,dx$

Definition of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

THEOREM 4.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

THEOREM 4.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions of kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

In Exercises 33–40, evaluate the integral using the following values.

$$\int_2^4 x^3\,dx = 60, \qquad \int_2^4 x\,dx = 6, \qquad \int_2^4 dx = 2$$

34. $\int_2^2 x^3\,dx$

In Exercises 33–40, evaluate the integral using the following values.

$$\int_2^4 x^3\,dx = 60, \qquad \int_2^4 x\,dx = 6, \qquad \int_2^4 dx = 2$$

38. $\int_2^4 (x^3 + 4)\,dx$

44. Given $\int_{-1}^{1} f(x)\,dx = 0$ and $\int_{0}^{1} f(x)\,dx = 5$, find

(a) $\int_{-1}^{0} f(x)\,dx$.

(b) $\int_{0}^{1} f(x)\,dx - \int_{-1}^{0} f(x)\,dx$.

(c) $\int_{-1}^{1} 3f(x)\,dx$.

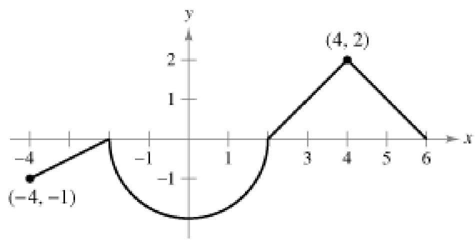
(d) $\int_{0}^{1} 3f(x)\,dx$.

45. ***Think About It*** The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.

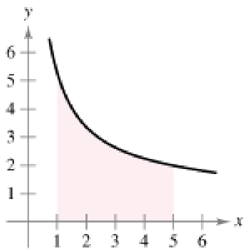
(a) $\int_0^2 f(x)\,dx$ (b) $\int_2^6 f(x)\,dx$

(c) $\int_{-4}^2 f(x)\,dx$ (d) $\int_{-4}^6 f(x)\,dx$

(e) $\int_{-4}^6 |f(x)|\,dx$ (f) $\int_{-4}^6 [f(x) + 2]\,dx$



In Exercises 47–50, use the figure to fill in the blank with the symbol <, >, or =.



47. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

48. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

68. Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$, where $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/3$, $x_3 = \pi$, and $x_4 = 2\pi$, and where $c_1 = \pi/6$, $c_2 = \pi/3$, $c_3 = 2\pi/3$, and $c_4 = 3\pi/2$.