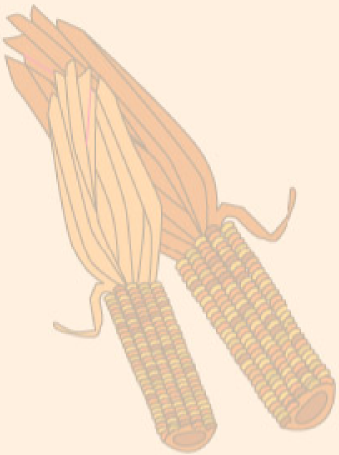
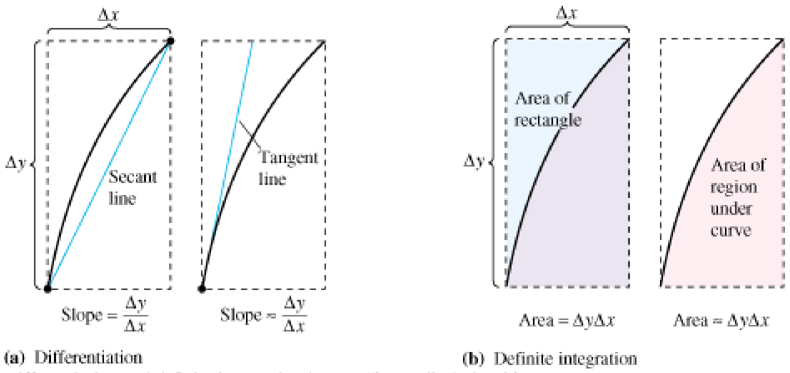
4.4 The Fundamental Theorem of Calculus



The Fundamental Theorem of Calculus



Differentiation and definite integration have an "inverse" relationship.

THEOREM 4.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Guidelines for Using the Fundamental Theorem of Calculus

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.**

1. $\int_0^{\pi} \frac{4}{x^2 + 1}\, dx$

2. $\int_0^{\pi} \cos x \, dx$

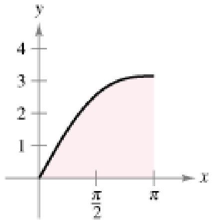
In Exercises 5–26, evaluate the definite integral of the algebraic function. Use a graphing utility to verify your result.

18. $\int_{1}^{8} \sqrt{\frac{2}{x}}\, dx$

24. $\int_{1}^{4} (3 - |x - 3|)\, dx$

In Exercises 35–40, determine the area of the indicated region.

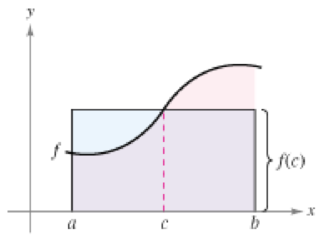
40. $y = x + \sin x$



In Exercises 41–44, find the area of the region bounded by the graphs of the equations.

44. $y = -x^2 + 3x, \qquad y = 0$

The Mean Value Theorem for Integrals



Mean value rectangle:

$$f(c)(b - a) = \int_a^b f(x)\,dx$$

THEOREM 4.10 Mean Value Theorem for Integrals

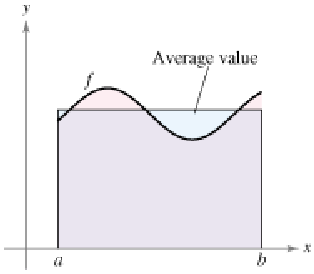
If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

In Exercises 45–48, find the value(s) of *c* guaranteed by the Mean Value Theorem for Integrals for the function over the indicated interval.

46. $f(x) = \frac{9}{x^3}$ $[1, 3]$

Average Value of a Function



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

In Exercises 49–52, find the average value of the function over the interval and all values of x in the interval for which the function equals its average value.

52. $f(x) = \cos x$ $\quad$ $[0, \pi/2]$

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

In Exercises 81–86, use the Second Fundamental Theorem of Calculus to find $F'(x)$.

82. $F(x) = \int_{1}^{x} \frac{t^2}{t^2 + 1}\, dt$

In Exercises 87–92, find $F'(x)$.

90. $F(x) = \int_{2}^{x^2} \frac{1}{t^3}\,dt$

92. $F(x) = \int_0^{x^2} \sin \theta^2 \, d\theta$

91. $F(x) = \int_0^{x^3} \sin t^2 \, dt$